SHARPNESS OF SOME PROPERTIES OF WIENER AMALGAM AND MODULATION SPACES

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ABSTRACT. We prove sharp estimates for the dilation operator $f(x) \longmapsto f(\lambda x)$, when acting on Wiener amalgam spaces $W(L^p, L^q)$. Scaling arguments are also used to prove the sharpness of the known convolution and pointwise relations for modulation spaces $M^{p,q}$, as well as the optimality of an estimate for the Schrödinger propagator on modulation spaces.

1. Introduction

Modulation and Wiener amalgam spaces have been introduced and used to measure the time-frequency concentration of functions and tempered distributions in the framework of time-frequency analysis [9, 10, 12, 13, 14, 15, 19]. Recently, these spaces have been employed to study boundedness properties of pseudodifferential operators (see, e.g., [4, 18, 20]), Fourier Integral operators (in particular, Fourier multipliers) [1, 7, 8] and wellposedness of solutions to PDE's (see, e.g., [2, 3, 5, 6, 21, 22, 23] and references therein).

In this paper we present new dilation properties for Wiener amalgam spaces and their optimality. Moreover, we prove the sharpness of the known convolution and pointwise estimates for modulation spaces.

To recall the definition of these spaces, we first introduce the translation and modulation operators, defined by $T_x f(t) = f(t-x)$ and $M_\xi f(t) = e^{2\pi i \xi t} f(t)$, $t, x, \xi \in \mathbb{R}^d$.

Wiener amalgam spaces [10, 12, 15]. Let $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be a test function. We will refer to g as a window function. Let B be either the Banach space $L^p(\mathbb{R}^d)$ or $\mathcal{F}L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. For any given function f which is locally in B (i.e. $gf \in B$, $\forall g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$), we set $f_B(x) = \|f T_x g\|_B$. The Wiener amalgam space $W(B, L^q)(\mathbb{R}^d)$ with local component B and global component $L^q(\mathbb{R}^d)$, $1 \leq q \leq \infty$, is defined as the space of all functions f locally in B such that $f_B \in L^q(\mathbb{R}^d)$. Endowed with the norm $\|f\|_{W(B,L^q)} = \|f_B\|_{L^q}$, $W(B, L^q)(\mathbb{R}^d)$ is a Banach space. Moreover, different choices of $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ generate the same space and yield equivalent norms. In fact, the space of admissible windows for the Wiener amalgam spaces $W(B, L^q)(\mathbb{R}^d)$ can

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be enlarged to the so-called Feichtinger algebra $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$. Recall that the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$.

Modulation spaces [10, 14]. For a fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is given by $V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle$, $x, \xi \in \mathbb{R}^d$.

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}(\mathbb{R}^{2d})$ (mixed-norm spaces). The norm on $M^{p,q}$ is given by

$$\|f\|_{M^{p,q}} := \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\xi)|^p\,dx\right)^{q/p} d\xi\right)^{1/q},$$

with obvious changes if $p = \infty$ or $q = \infty$. If $p = q$, we write M^p instead of $M^{p,p}$. The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space, whose definition is independent of the choice of the window g. Different non-zero windows $g \in M^1$ yield equivalent norms on $M^{p,q}$. This property will be crucial in the sequel, because we will choose a suitable window g in estimates of the $M^{p,q}$-norm. Within the class of modulation spaces, one finds several standard function spaces, for instance $M^2 = L^2$, $M^1 = W(\mathcal{F}L^1, L^1)$ and, using weighted versions, one also finds certain Sobolev spaces and Shubin-Sobolev spaces ([4, 14]). The relationship between modulation and Wiener amalgam spaces is expressed by the following result:

The Fourier transform establishes an isomorphism $\mathcal{F} : M^{p,q} \to W(\mathcal{F}L^p, L^q)$.

Consequently, convolution properties of modulation spaces can be translated into pointwise multiplication properties of Wiener amalgam spaces.

Let us now turn to the topic of the present paper. The importance of the dilation operator

$$f(x) \longmapsto f(\lambda x), \quad \lambda > 0,$$

in classical analysis is well-known. For example, in most estimates arising in classical Harmonic Analysis (e.g. the Hölder, Young and Hausdorff-Young inequalities) as well as in Partial Differential Equations (e.g. Sobolev embeddings, Strichartz estimates) scaling arguments yield the constraints that the Lebesque exponents must satisfy for the corresponding inequalities to hold.

When dealing with modulation or Wiener amalgam spaces, the situation becomes more subtle. In fact, the corresponding norms are not 'homogeneous' with respect to the scaling. Basically, this is due to the fact that, for example, in $W(L^p, L^q)$ the two spaces L^p, L^q display different scaling if $p \neq q$. Obtaining sharp estimates (in terms of λ) for the dilation operator norm, when acting on such spaces, is therefore a non trivial problem. This study was widely carried out in [17] (see also [5, 20]) in the case of modulation spaces $M^{p,q}$. The estimates obtained in [17] turned out to be a fundamental tool for embedding problems of modulation spaces into Besov spaces

(see also [22]), and for boundedness of pseudodifferential operators of type (ρ,δ) on modulation spaces ([18]). Let us highlight that this type of arguments allowed us to prove the sharpness of some Strichartz estimates in the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$ in [6]. Finally, they were also used in [8] to prove sharp boundedness properties of Hörmader's type Fourier integral operators on $\mathcal{F}L^p$ and modulation spaces.

Let us point out that an investigation of the dilation operator on $W(C, L^1)$ (C being the space of continuous functions) had already appeared in [13].

The first result of this note (Proposition 2.1 below) provides *sharp* upper and lower bounds for the operator norm of the dilation operator on the Wiener amalgam spaces $W(L^p, L^q)$.

Differently from what one could ingenuously expect, it does not happen that the exponent p alone has its influence when $\lambda \to +\infty$, whereas the exponent q when $\lambda \to 0$.

Then, as for the classical function spaces, scaling arguments are employed to prove the sharpness of the known convolution, inclusion and pointwise moltiplication relations for modulation spaces. This is precisely the topic studied in Section 3. To chase this goal, we do not use the bounds obtained in [17], that would give weaker constraints than the optimal ones. Instead, the sharp results are the issues of explicit computations involving dilation properties of Gaussian functions.

Finally, we observe that these techniques can be applied to prove the sharpness of estimates arising in PDEs. As an example, in Section 4 we prove the optimality of an estimate for the Schrödinger propagator, recently obtained in [22] (see also [1]).

2. Dilation properties of Wiener amalgam spaces

In this section we study the dilation properties of Wiener amalgam spaces $W(L^p, L^q)$, $1 \leq p, q \leq \infty$. First, recall the following complex interpolation result [9].

Lemma 2.1. *Let B_0, B_1, be local components of Wiener amalgam spaces, as in the Introduction. Then, for $1 \leq q_0, q_1 \leq \infty$, with $q_0 < \infty$ or $q_1 < \infty$, and $0 < \theta < 1$, we have*

$$[W(B_0, L^{q_0}), W(B_1, L^{q_1})]_{[\theta]} = W\left([B_0, B_1]_{[\theta]}, L^{q_\theta}\right),$$

with $q_\theta = (1-\theta)/q_0 + \theta/q_1$.

For $\lambda > 0$, we set $f_\lambda(x) = f(\lambda x)$.

Proposition 2.1. *For $1 \leq p, q \leq \infty$,*

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{-d\max\left\{\frac{1}{p},\frac{1}{q}\right\}} \|f\|_{W(L^p,L^q)}, \quad \forall\, 0 < \lambda \leq 1, \tag{1}$$

and

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{-d\min\left\{\frac{1}{p},\frac{1}{q}\right\}} \|f\|_{W(L^p,L^q)}, \quad \forall \lambda \geq 1. \tag{2}$$

Also, we have

$$\|f_\lambda\|_{W(L^p,L^q)} \gtrsim \lambda^{-d\min\left\{\frac{1}{p},\frac{1}{q}\right\}}\|f\|_{W(L^p,L^q)}, \quad \forall\, 0<\lambda\leq 1, \tag{3}$$

and

$$\|f_\lambda\|_{W(L^p,L^q)} \gtrsim \lambda^{-d\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\|f\|_{W(L^p,L^q)}, \quad \forall \lambda\geq 1. \tag{4}$$

We first prove the following weaker estimates.

Lemma 2.2. *For $1\leq p,q\leq\infty$,*

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{-d\left(\frac{1}{p}+\frac{1}{q}\right)}\|f\|_{W(L^p,L^q)}, \quad \forall\, 0<\lambda\leq 1, \tag{5}$$

and

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{d\left(1-\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{W(L^p,L^q)}, \quad \forall \lambda\geq 1. \tag{6}$$

Proof. To compute the Wiener norm, we choose the window function $g=\chi_{B(0,1)}$, the characteristic function of the ball $B(0,1)$. Then,

$$\begin{aligned}\|f_\lambda\|_{W(L^p,L^q)} &\asymp \|\|f(\lambda t)g(t-x)\|_{L^p_t}\|_{L^q_x}\\ &= \lambda^{-\frac{d}{p}}\|\|f(t)g_{1/\lambda}(t-\lambda x)\|_{L^p_t}\|_{L^q_x}\\ &= \lambda^{-d\left(\frac{1}{p}+\frac{1}{q}\right)}\|\|f(t)g_{1/\lambda}(t-x)\|_{L^p_t}\|_{L^q_x}.\end{aligned}$$

If $0<\lambda\leq 1$, the window function g fulfills $g_{1/\lambda}(y)\leq g(y)$, and (5) follows.
To prove (6), we argue by duality. Indeed, if $\lambda\geq 1$, relation (5), applied to the pair (p',q'), yields

$$\begin{aligned}\|f_\lambda\|_{W(L^p,L^q)} &= \sup_{\|g\|_{W(L^{p'},L^{q'})}=1}|\langle f_\lambda,g\rangle|\\ &= \sup_{\|g\|_{W(L^{p'},L^{q'})}=1}\lambda^{-d}|\langle f,g_{1/\lambda}\rangle|\\ &\leq \lambda^{-d}\sup_{\|g\|_{W(L^{p'},L^{q'})}=1}\|f\|_{W(L^p,L^q)}\|g_{1/\lambda}\|_{W(L^{p'},L^{q'})}\\ &\lesssim \lambda^{-d}\lambda^{d\left(\frac{1}{p'}+\frac{1}{q'}\right)}\|f\|_{W(L^p,L^q)}.\end{aligned}$$

□

Proof of Proposition 2.1. We first prove (1) and (2) when $p=\infty$. We see at once that (1) coincides with (5) when $p=\infty$. On the other hand, (2) for $p=\infty$ follows by complex interpolation (Lemma 2.1) from (6) with $(p,q)=(\infty,1)$, i.e.,

$$\|f_\lambda\|_{W(L^\infty,L^1)}\lesssim\|f\|_{W(L^\infty,L^1)}$$

and the trivial estimate

$$\|f_\lambda\|_{W(L^\infty,L^\infty)}\asymp\|f_\lambda\|_{L^\infty}=\|f\|_{L^\infty}\asymp\|f\|_{W(L^\infty,L^\infty)}.$$

Since the estimates (1) and (2) also hold for $p = q$ (because $W(L^p, L^p) = L^p$ with equivalent norms), by interpolation with the case $p = \infty$, $1 \le q \le \infty$, we see that they hold for any pair (p, q), with $1 \le q \le p \le \infty$. When $p < q$ they follow by duality arguments as in the proof of (6).

Finally, (3) and (4) follow at once from (2) and (1), respectively, applied to the function $f_{1/\lambda}$.

□

We now show that the result above is sharp.

Proposition 2.2. (Sharpness of (1) and (2)).
(i) Suppose that, for some $\alpha \in \mathbb{R}$*,*

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^\alpha \|f\|_{W(L^p,L^q)}, \quad \forall\, 0 < \lambda \le 1. \tag{7}$$

Then

$$\alpha \le -d \max\left\{\frac{1}{p}, \frac{1}{q}\right\}. \tag{8}$$

(ii) Suppose that, for some $\alpha \in \mathbb{R}$*,*

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^\alpha \|f\|_{W(L^p,L^q)}, \quad \forall \lambda \ge 1. \tag{9}$$

Then

$$\alpha \ge -d \min\left\{\frac{1}{p}, \frac{1}{q}\right\}. \tag{10}$$

This also shows the sharpness of the estimates (3) and (4), since they are equivalent to (2) and (1), respectively.

Proof. (i) First, consider the case $p \ge q$. We have $W(L^p, L^q) \hookrightarrow W(L^q, L^q) = L^q$. Hence

$$\lambda^{-\frac{d}{q}} \|f\|_{L^q} = \|f_\lambda\|_{L^q} \lesssim \|f_\lambda\|_{W(L^p,L^q)}.$$

Combining this estimate with (7) and letting $\lambda \to 0^+$, we obtain $\alpha \le -d/q$.

Assume now $p < q$. It suffices to verify that, for every $\epsilon > 0$, there exists $f \in W(L^p, L^q)$ such that

$$\|f_\lambda\|_{W(L^p,L^q)} \ge C \lambda^{-\frac{d}{p}+\epsilon}. \tag{11}$$

We study the case of dimension $d = 1$. The general case follows by tensor products of functions of one variable. To this end, we choose

$$f(t) = \begin{cases} |t|^{-\frac{1}{p}+\epsilon} & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1. \end{cases}$$

Observe that $f \in W(L^p, L^1) \hookrightarrow W(L^p, L^q)$, for every $1 \leq q \leq \infty$, and

$$f(\lambda t) = \lambda^{-\frac{1}{p}+\epsilon} f(t), \quad \text{for } |t| \leq \frac{1}{\lambda}. \tag{12}$$

Now, take $g = \chi_{B(0,1)}$ as window function. Of course,

$$\begin{aligned} \|f_\lambda\|_{W(L^p,L^q)} &= \left(\int \|f_\lambda T_y g\|_{L^p}^q dy \right)^{1/q} \\ &\geq \left(\int_{B(0,1)} \|f_\lambda T_y g\|_{L^p}^q dy \right)^{1/q}. \end{aligned}$$

By using (12) and the choice $g = \chi_{B(0,1)}$, for $\lambda \leq 1/2$ the last expression is estimated from below by

$$\geq \lambda^{-1/p+\epsilon} \left(\int_{B(0,1)} \|f T_y g\|_{L^p}^q dy \right)^{1/q},$$

that is (11).

(ii) Again, we first consider the case $p \geq q$, namely $q' \geq p'$. Then $L^{p'} = W(L^{p'}, L^{p'}) \hookrightarrow W(L^{p'}, L^{q'})$. Hence,

$$\|f_\lambda\|_{W(L^p,L^q)} = \sup_{\|g\|_{W(L^{p'},L^{q'})}=1} |\langle f_\lambda, g\rangle| \gtrsim \sup_{\|g\|_{L^{p'}}=1} |\langle f_\lambda, g\rangle| = \|f_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}} \|f\|_{L^p}.$$

Combining this estimate with (9) and letting $\lambda \longrightarrow +\infty$, we obtain $\alpha \geq -\frac{d}{q}$.

Suppose now $p < q$. As before it suffices to prove, in dimension $d = 1$, that for every $\epsilon > 0$ there exists a function $f \in W(L^p, L^q)$ such that

$$\|f_\lambda\|_{W(L^p,L^q)} \geq C\lambda^{-\frac{1}{q}-\epsilon}.$$

Therefore, choose

$$f(t) = \begin{cases} |t|^{-\frac{1}{q}-\epsilon} & \text{for } |t| \geq 1 \\ 0 & \text{for } |t| < 1. \end{cases}$$

Then $f \in W(L^\infty, L^q) \hookrightarrow W(L^p, L^q)$, for every $1 \leq p \leq \infty$, and

$$f(\lambda t) = \lambda^{-\frac{1}{q}-\epsilon} f(t), \quad \text{for } |t| \geq \frac{1}{\lambda}. \tag{13}$$

Again, choose $g = \chi_{B(0,1)}$ as window function. We have

$$\|f_\lambda\|_{W(L^p,L^q)} \geq \left(\int_{B(0,2)} \|f_\lambda T_y g\|_{L^p}^q dy \right)^{1/q}.$$

By using (13) and the choice $g = \chi_{B(0,1)}$, for $\lambda \geq 1$ the last expression is

$$\geq \lambda^{-1/q-\epsilon} \left(\int_{B(0,2)} \|f T_y g\|_{L^p}^q dy \right)^{1/q},$$

which concludes the proof of (ii). $\square$

3. Convolution, inclusion and multiplication relations for modulation spaces

In this section we study the optimality of the convolution, inclusion and pointwise multiplication relations for modulation spaces. We need some preliminary results.

If one chooses the Gaussian $e^{-\pi|x|^2}$ as window function to compute Wiener amalgam norms, then an easy computation (see e.g. [6, Lemma 5.3]) yields the result below.

Lemma 3.1. *For $a, b \in \mathbb{R}$, $a > 0$, set $\mathcal{G}_{(a+ib)}(x) = (a+ib)^{-d/2} e^{-\frac{\pi|x|^2}{a+ib}}$. Then, for every $1 \leq p, q \leq \infty$,*

$$\|\mathcal{G}_{(a+ib)}\|_{W(\mathcal{F}L^p, L^q)} = \frac{((a+1)^2 + b^2)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{2}\right)}}{p^{\frac{d}{2p}} (aq)^{\frac{d}{2q}} \left(a(a+1) + b^2\right)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}}. \tag{14}$$

For tempered distributions compactly supported either in time or in frequency, the $M^{p,q}$-norm is equivalent to the $\mathcal{F}L^q$-norm or L^p-norm, respectively. This result is well-known ([11, 12, 16]). For the sake of completeness we provide an outline of the proof.

Lemma 3.2. *Let $1 \leq p, q \leq \infty$.*
(i) For every $u \in \mathcal{S}'(\mathbb{R}^d)$, supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in M^{p,q} \Leftrightarrow u \in \mathcal{F}L^q$, and

$$C_K^{-1} \|u\|_{M^{p,q}} \leq \|u\|_{\mathcal{F}L^q} \leq C_K \|u\|_{M^{p,q}}, \tag{15}$$

where $C_K > 0$ depends only on K.
(ii) For every $u \in \mathcal{S}'(\mathbb{R}^d)$, whose Fourier transform is supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in M^{p,q} \Leftrightarrow u \in L^p$, and

$$C_K^{-1} \|u\|_{M^{p,q}} \leq \|u\|_{L^p} \leq C_K \|u\|_{M^{p,q}}, \tag{16}$$

where $C_K > 0$ depends only on K.

Proof. (i) It is detailed in [16, Lemma 1].
(ii) It is well-known (see e.g. [19]) that

$$\|u\|_{M^{p,q}} \asymp \left(\sum_{k \in \mathbb{Z}^d} \|\nu(D-k)u\|_{L^p}^q \right)^{1/q},$$

where ν is a test function satisfying $\sum_{k \in \mathbb{Z}^d} \nu(\xi - k) \equiv 1$. Now, if $\hat{u}$ has compact support, the above sum is finite and one deduces at once the first estimate in (16), since the multipliers $\nu(D-k)$ are (uniformly) bounded on L^p. To obtain the second

estimate in (16), we write $u = \sum_{k\in\mathbb{Z}^d} \nu(D-k)u$, then apply the triangle inequality and the finiteness of the sum over k again. ∎

Now, we turn our attention to the sharpness of the convolution properties for modulation spaces.

Proposition 3.1. *Let* $1 \leq p, q, p_1, p_2, q_1, q_2 \leq \infty$. *Then*

$$\|f * g\|_{M^{p,q}} \lesssim \|f\|_{M^{p_1,q_1}} \|g\|_{M^{p_2,q_2}} \tag{17}$$

if and only if the following indices' relations hold true:

$$\frac{1}{p} + 1 \leq \frac{1}{p_1} + \frac{1}{p_2}, \tag{18}$$

and

$$\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}. \tag{19}$$

Proof. Sufficiency. The inclusion relations (17), were proved in [4, 20]. There the indices' relations (18) and (19) were shown with the equalities. The inequalities follow by the inclusion relations $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$ for $p_1 \leq p_2$ and $q_1 \leq q_2$ ([10, 14]).
Necessity. We consider the family of Gaussians $\varphi_\lambda(x) := e^{-\pi\lambda|x|^2}$, for $\lambda > 0$. Obviously, $\varphi_\lambda \in \mathcal{S}(\mathbb{R}^d) \subset M^{p,q}(\mathbb{R}^d)$, for every $1 \leq p, q \leq \infty$. Since $\|f\|_{M^{p,q}} \asymp \|\hat{f}\|_{W(\mathcal{F}L^p,L^q)}$ and $\hat{\varphi}_\lambda = \lambda^{-d/2}\varphi_{1/\lambda}$, Lemma 3.1 yields:

$$\|\varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-d/2}\|\varphi_{1/\lambda}\|_{W(\mathcal{F}L^p,L^q)} \asymp \|\mathcal{G}_{(\lambda)}\|_{W(\mathcal{F}L^p,L^q)} \asymp \frac{(\lambda+1)^{d(1/p-1/2)}}{\lambda^{d/(2q)}(\lambda^2+\lambda)^{(1/p-1/q)d/2}}. \tag{20}$$

A straightforward calculation shows that $(\varphi_\lambda * \varphi_\lambda)(x) = (2\lambda)^{-\frac{d}{2}}\varphi_{\lambda/2}(x)$. Hence, using (20), we obtain

$$\|\varphi_\lambda * \varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-(1+1/p)d/2}, \quad \text{for } \lambda \to 0^+. \tag{21}$$

Using (20) again, we also obtain

$$\|\varphi_\lambda\|_{M^{p_i,q_i}} \asymp \lambda^{-\frac{d}{2p_i}}, \quad i = 1, 2, \qquad \text{for } \lambda \to 0^+. \tag{22}$$

Substituting in (17), we obtain (18). The relation (19) can be obtained similarly. Indeed, the estimate (20) gives, for $\lambda \to +\infty$,

$$\|\varphi_\lambda * \varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-d(1-\frac{1}{2q})}, \quad \|\varphi_\lambda\|_{M^{p_i,q_i}} \asymp \lambda^{-\frac{d}{2}(1-\frac{1}{q_i})}, \quad i = 1, 2,$$

and, using (17) again, the relation (19) follows.

An alternative proof of the necessary conditions (18) and (19) is provided by Lemma 3.2. Precisely, to prove (19), consider two compactly supported smooth functions f, g and their scaling $f_\lambda(x) = f(\lambda x)$, $g_\lambda(x) = g(\lambda x)$, with $\lambda \geq 1$. Since $\lambda \geq 1$, f_λ and g_λ (and therefore $f_\lambda * g_\lambda$) are all supported in a compact subset

K, independent of λ. By Lemma 3.2, (i), the bilinear estimate (17) for f_λ and g_λ becomes

$$\|f_\lambda * g_\lambda\|_{\mathcal{F}L^q} \lesssim \|f_\lambda\|_{\mathcal{F}L^{q_1}} \|g_\lambda\|_{\mathcal{F}L^{q_2}}.$$

Using $f_\lambda * g_\lambda = \lambda^{-d}(f*g)_\lambda$, the dilation property for $\mathcal{F}L^q$ spaces: $\|h(\lambda \cdot)\|_{\mathcal{F}L^q} = \lambda^{-\frac{d}{q'}}\|h\|_{\mathcal{F}L^q}$, and letting $\lambda \to +\infty$, we obtain (19).

In order to prove (18), one argues similarly. Here the functions f, g have Fourier transforms $\hat{f}, \hat{g}$ compactly supported and the scale λ satisfies $0 < \lambda \leq 1$. By Lemma 3.2, (ii), the estimate (17) becomes

$$\|f_\lambda * g_\lambda\|_{L^p} \lesssim \|f_\lambda\|_{L^{p_1}} \|g_\lambda\|_{L^{p_2}}.$$

Using $f_\lambda * g_\lambda = \lambda^{-d}(f*g)_\lambda$, the dilation property $\|h(\lambda\cdot)\|_{L^p} = \lambda^{-d/p}\|h\|_{L^p}$, and letting $\lambda \to 0^+$, we prove (18). □

The family of Gaussians φ_λ provides an alternative proof for the sharpness of the inclusion relation for modulation spaces, already obtained by the inclusion relations for the sequence spaces $\ell^{p,q}$, via the norm equivalence $\|f\|_{M^{p,q}} \asymp \|\langle f, T_m M_n g\rangle\|_{\ell^{p,q}}$, with $\{T_m M_n g\}$ being a Gabor frame (see, e.g., [14, Theorem 13.6.1]).

Proposition 3.2. *Let* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. *Then,*

$$\|f\|_{M^{p_2,q_2}} \lesssim \|f\|_{M^{p_1,q_1}} \tag{23}$$

if and only if the following indices' relation holds

$$p_1 \leq p_2 \quad \textit{and} \quad q_1 \leq q_2. \tag{24}$$

Proof. We show the necessity of (24). Let $\varphi_\lambda(x) = e^{-\pi\lambda|x|^2}$, $\lambda > 0$. From the proof of Proposition 3.1,

$$\|\varphi_\lambda\|_{M^{p_i,q_i}} \asymp \lambda^{-\frac{d}{2p_i}}, \quad \text{for } \lambda \to 0^+ \quad \text{and} \quad \|\varphi_\lambda\|_{M^{p_i,q_i}} \asymp \lambda^{-\frac{d}{2}(1-\frac{1}{q_i})}, \quad \text{for } \lambda \to +\infty. \tag{25}$$

Hence, for (23) to be satisfied it must be

$$\lambda^{-\frac{d}{2p_2}} \lesssim \lambda^{-\frac{d}{2p_1}}, \quad \text{for } \lambda \to 0^+ \quad \text{and} \quad \lambda^{-\frac{d}{2}(1-\frac{1}{q_2})} \lesssim \lambda^{-\frac{d}{2}(1-\frac{1}{q_1})}, \quad \text{for } \lambda \to +\infty,$$

that give the indices' relations in (24). □

In what follows we study the pointwise multiplication operator in modulation spaces (which is equivalent to studing the convolution operator for the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$).

Proposition 3.3. *Let* $1 \leq p, q, p_1, p_2, q_1, q_2 \leq \infty$. *Then*

$$\|fg\|_{M^{p,q}} \lesssim \|f\|_{M^{p_1,q_1}} \|g\|_{M^{p_2,q_2}} \tag{26}$$

if and only if the following indices' relations hold true:

$$\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}, \tag{27}$$

and

$$\frac{1}{q} + 1 \leq \frac{1}{q_1} + \frac{1}{q_2}. \tag{28}$$

Proof. The sufficiency can be found in [10] (see also [23]). For the necessity of the conditions (27) and (28) we test the estimate (26) on the Gaussians $f(x) = g(x) = \varphi_\lambda(x) = e^{-\lambda\pi|x|^2}$. We observe that $\varphi_\lambda\varphi_\lambda = \varphi_{2\lambda}$. Hence by applying (25) and substituting in (26), relation (28) follows by letting $\lambda \to 0^+$, whereas (27) follows by letting $\lambda \to +\infty$. □

4. An estimate for the Schrödinger propagator

Consider the Fourier multiplier $e^{it\Delta}$, with symbol $e^{-it|2\pi\xi|^2}$, i.e.,

$$\left(e^{it\Delta}u_0\right)(x) = \frac{1}{(4\pi it)^{d/2}} \int e^{i\frac{|x-y|^2}{4t}} u_0(y)\, dy.$$

It is shown in [22, Proposition 4.1] that, given $2 \leq p < \infty$, $1/p + 1/p' = 1$, $1 \leq q \leq \infty$,

$$\|e^{it\Delta}u_0\|_{M^{p,q}} \lesssim (1+|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{M^{p',q}}. \tag{29}$$

(Similar estimates were obtained in [1]). We now show that the condition $p \geq 2$ is necessary in (29), and the decay at infinity is optimal.

Proposition 4.1. (Sharpness of (29)). *Suppose that, for some fixed $t_0 \in \mathbb{R}$, $1 \leq p, q \leq \infty$, $C > 0$, the following estimate holds:*

$$\|e^{it_0\Delta}u_0\|_{M^{p,q}} \leq C\|u_0\|_{M^{p',q}}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^d). \tag{30}$$

Then $p \geq 2$.

Assume now that, for some $\alpha \in \mathbb{R}$, $C > 0$, $M > 0$, $1 \leq \gamma, \delta \leq \infty$, $1 \leq p, q \leq \infty$, the estimate

$$\|e^{it\Delta}u_0\|_{M^{p,q}} \leq Ct^\alpha \|u_0\|_{M^{\gamma,\delta}}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^d), \tag{31}$$

holds for every $t > M$. Then

$$\alpha \geq -d\left(\frac{1}{2} - \frac{1}{p}\right). \tag{32}$$

Proof. We consider the family of initial data $u_0(\lambda x) = e^{-\pi\lambda^2|x|^2}$, $\lambda > 0$. A direct computation shows that the corresponding solutions are

$$u(\lambda^2 t, \lambda x) = (1 + 4\pi i t\lambda^2)^{-d/2} e^{-\frac{\pi\lambda^2|x|^2}{1+4\pi i t\lambda^2}} \tag{33}$$
$$= \lambda^{-d} \mathcal{G}_{(\lambda^{-2}+4\pi i t)}(x),$$

where we used the notation in Lemma 3.1. It follows from (14) that

$$\|u_0(\lambda\cdot)\|_{M^{p',q}} \asymp \lambda^{-d}\|\hat{u}_0(\lambda^{-1}\cdot)\|_{W(\mathcal{F}L^{p'},L^q)} = \|\mathcal{G}_{(\lambda^2)}\|_{W(\mathcal{F}L^{p'},L^q)} \asymp \lambda^{-\frac{d}{p'}}, \quad \text{as } \lambda \to 0^+. \tag{34}$$

On the other hand, by (33),

$$\|u(\lambda^2 t, \lambda\,\cdot)\|_{M^{p,q}} \asymp \|\mathcal{F}\left(u(\lambda^2 t, \lambda\,\cdot)\right)\|_{W(\mathcal{F}L^p,L^q)} \asymp \lambda^{-d}(a^2+b^2)^{\frac{d}{4}}\|\mathcal{G}_{(a+ib)}\|_{W(\mathcal{F}L^p,L^q)}, \tag{35}$$

where

$$a = \frac{\lambda^{-2}}{\lambda^{-4} + (4\pi t)^2}, \quad b = -\frac{4\pi t}{\lambda^{-4} + (4\pi t)^2}.$$

Hence, for fixed $t = t_0$, (14) gives

$$\|u(\lambda^2 t_0, \lambda\,\cdot)\|_{M^{p,q}} \asymp \lambda^{-\frac{d}{p}}, \quad \text{as } \lambda \to 0^+. \tag{36}$$

Estimates (34), (36) and (30) yield $-\frac{d}{p} \geq -\frac{d}{p'}$, namely $p \geq 2$.

Choosing $\lambda = 1$ in (35) and using (14), we obtain

$$\|u(t,\cdot)\|_{M^{p,q}} \asymp t^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}, \quad \text{as } t \to +\infty.$$

This shows that (32) is necessary for (31) to hold. ∎

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